

A Model of Unified Gauge Interactions

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Abstract

Linear spinor fields are a generalization of the Dirac field that have direct correspondence with the known physics of fermions, inherent causality properties in their most fundamental constructions, and positive mass eigenvalues for all particle types. The algebra of the generators for infinitesimal transformations of these fields directly constructs the Minkowski metric *within* the internal group space as a consequence of non-vanishing commutation relations between generators that carry space-time indexes. In addition, the generators have a fundamental matrix representation that includes Lorentz transformations within a group that unifies internal gauge symmetries generated by a set of hermitian generators for $SU(3) \times SU(2) \times U(1)$, and nothing else. The construction of linearly independent internal $SU(3)$ and $SU(2)$ symmetry groups necessarily involves the mixing of three generations of the mass eigenstates labeling the (massive) representations of the linear spinor fields. The group algebra also provides a mechanism for the dynamic mixing of massless particles of differing “transverse mass” eigenvalues conjugate to the affine parameter labeling translations along their light-like trajectories. The inclusion of a transverse mass generator is necessary for group closure of the extended Poincare algebra, but its eigenvalue must vanish for massive particle representations. A unified set of space-time group transformation operations along with internal gauge group symmetry operations for linear spinor fields will be demonstrated in this paper.

1 Introduction

The Dirac equation utilizes a matrix algebra to construct a field equation that is linear in the quantum operators for 4-momentum. Since inversion of linear operations is straightforward, the properties of evolution dynamics described using such linear operations on quantum states have direct interpretations (e.g. towards constructions of resolvents or propagators)[1][2][3]. The Dirac formulation can be extended to generally require that the form $\hat{\Gamma}^\mu \hat{P}_\mu$ be a Lorentz scalar operation, resulting in a spinor field equation of the form

$$\mathbf{\Gamma}^\beta \cdot \frac{\hbar}{i} \frac{\partial}{\partial x^\beta} \hat{\Psi}_{(\gamma)}^{(\Gamma)}(\vec{x}) = -(\gamma)mc \hat{\Psi}_{(\gamma)}^{(\Gamma)}(\vec{x}), \quad (1.1)$$

where m is positive for all particle types, and $\mathbf{\Gamma}^\beta$ are finite dimensional matrix representations of the operators $\hat{\Gamma}^\beta$. For massive particles, the particle type label (γ) is just the particular eigenvalue of the hermitian matrix $\mathbf{\Gamma}^0$. In the $\Gamma = \frac{1}{2}$ representation, the matrix form of the operators $\mathbf{\Gamma}^\beta = \frac{1}{2}\gamma^\beta$ are one half of the Dirac matrices[4][5], and the particle type label takes values $(\gamma) = \pm\frac{1}{2}$, eliminating the need for any filled “Dirac sea” of negative energy states.

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2 An Extension of the Lorentz and Poincare Groups

An extension of the Lorentz group can be defined using the algebra

$$[\Gamma^0, \Gamma^k] = i K_k, \quad (2.2)$$

$$[\Gamma^0, J_k] = 0, \quad (2.3)$$

$$[\Gamma^0, K_k] = -i \Gamma^k, \quad (2.4)$$

$$[\Gamma^j, \Gamma^k] = -i \epsilon_{jkm} J_m, \quad (2.5)$$

$$[\Gamma^j, J_k] = i \epsilon_{jkm} \Gamma^m, \quad (2.6)$$

$$[\Gamma^j, K_k] = -i \delta_{jk} \Gamma^0, \quad (2.7)$$

which has a Casimir operator $C_\Gamma = \underline{J} \cdot \underline{J} - \underline{K} \cdot \underline{K} + \Gamma^0 \Gamma^0 - \underline{\Gamma} \cdot \underline{\Gamma}$ that commutes with all generators of the group. The operators C_Γ , Γ^0 , and J_z have been chosen as the set of mutually commuting operators for the construction of the finite dimensional representations.

The matrices corresponding to $\Gamma = \frac{1}{2}$ (the fundamental representation) have dimensionality $N_{\frac{1}{2}} = 4$, and will be expressed in terms of the Pauli spin matrices σ_j as demonstrated below:

$$\begin{aligned} \mathbf{\Gamma}^0 &= \frac{1}{2} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} & \mathbf{J}_j &= \frac{1}{2} \begin{pmatrix} \sigma_j & \mathbf{0} \\ \mathbf{0} & \sigma_j \end{pmatrix} \\ \mathbf{\Gamma}^j &= \frac{1}{2} \begin{pmatrix} \mathbf{0} & \sigma_j \\ -\sigma_j & \mathbf{0} \end{pmatrix} & \mathbf{K}_j &= -\frac{i}{2} \begin{pmatrix} \mathbf{0} & \sigma_j \\ \sigma_j & \mathbf{0} \end{pmatrix} \end{aligned} \quad (2.8)$$

A representation for the $\Gamma = 1$ matrices can be found in Appendix D.2.1 of reference [1].

The inclusion of space-time translations into the group algebra must result in a self-consistent closed set of generators. The 4-momentum operators together with the extended Lorentz group operators do not produce a closed group structure, due to Jacobi relations of the type $[\hat{P}_j, [\hat{\Gamma}^0, \hat{\Gamma}^k]]$. An additional momentum-like operator, which will be labeled \mathcal{M}_T , must be introduced, resulting in additional non-vanishing commutators:

$$[J_j, P_k] = i\hbar \epsilon_{jkm} P_m, \quad (2.9)$$

$$[K_j, P_0] = -i\hbar P_j, \quad (2.10)$$

$$[K_j, P_k] = -i\hbar \delta_{jk} P_0, \quad (2.11)$$

$$[\Gamma^\mu, P_\nu] = i \delta_\nu^\mu \mathcal{M}_T c, \quad (2.12)$$

$$[\Gamma^\mu, \mathcal{M}_T] = \frac{i}{c} \eta^{\mu\nu} P_\nu, \quad (2.13)$$

The final two relations extend the Poincare algebra as necessary to close the algebra.

Linearity of the operator $\Gamma^\mu P_\mu$ in the energy-momentum generators is quite useful in developing the field equations defining linear spinor fields. Important commutation relations of this operator are given

below:

$$[J_k, \Gamma^\mu P_\mu] = 0 \quad (2.14)$$

$$[K_k, \Gamma^\mu P_\mu] = 0 \quad (2.15)$$

$$[P_\beta, \Gamma^\mu P_\mu] = -i\mathcal{M}_T P_\beta \quad (2.16)$$

$$[\mathcal{M}_T, \Gamma^\mu P_\mu] = -i\eta^{\beta\nu} P_\beta P_\nu \quad (2.17)$$

From (2.17), the operator $\hat{\mathcal{M}}_T$ only commutes with $\Gamma^\mu P_\mu$ for massless particles, while from (2.16) the 4-momentum operator only commutes with $\Gamma^\mu P_\mu$ for those states with eigenvalues of $\hat{\mathcal{M}}_T$ that vanish. For massless states $m_T \neq 0$, the operator $\hat{\mathcal{M}}_T$ is the generator for translations along the affine parameter labeling the light-like trajectory of the massless particle.

2.1 Unitary massive particle states

A Casimir operator for the complete extended Poincare (EP) group whose eigenvalues label irreducible particle states can be constructed from the Lorentz invariants

$$\mathcal{C}_m \equiv \mathcal{M}_T^2 c^2 - \eta^{\beta\nu} P_\beta P_\nu. \quad (2.18)$$

This form suggests that the hermitian operator \mathcal{M}_T be referred to as a *transverse mass* parameter of the state. The quantum (standard) state vectors labeled using mutually commuting operators satisfy

$$\begin{aligned} \hat{\mathcal{C}}_m |m, \Gamma, \gamma, J, s_z\rangle &= m^2 c^2 |m, \Gamma, \gamma, J, s_z\rangle, \\ \hat{\mathcal{C}}_\Gamma |m, \Gamma, \gamma, J, s_z\rangle &= 2\Gamma(\Gamma + 2) |m, \Gamma, \gamma, J, s_z\rangle, \\ \hat{\Gamma}^0 |m, \Gamma, \gamma, J, s_z\rangle &= \gamma |m, \Gamma, \gamma, J, s_z\rangle, \\ \hat{J}^2 |m, \Gamma, \gamma, J, s_z\rangle &= J(J + 1)\hbar^2 |m, \Gamma, \gamma, J, s_z\rangle, \\ \hat{J}_z |m, \Gamma, \gamma, J, s_z\rangle &= s_z \hbar |m, \Gamma, \gamma, J, s_z\rangle, \end{aligned} \quad (2.19)$$

where Γ is an integral or half-integral label of the representation of the extended Lorentz group, and J , which has the same integral signature as Γ , labels the internal angular momentum representation of the state. Unitary representations of general momentum states are generated via boosting standard states satisfying (2.19). For a more complete treatment of the algebra, symmetries, and causality properties of linear spinor fields, the reader is invited to examine sections 4.3 and 4.4 in reference [1].

2.2 Development of group metric on space-time indexes

For a general algebra satisfying $[\hat{G}_r, \hat{G}_s] = -i \sum_m (c_s)_r^m \hat{G}_m$, the adjoint representation expressed in terms of the structure constants defines a group metric η_{ab} given by

$$\eta_{ab} \equiv \sum_{s,r} (c_a)_r^s (c_b)_s^r. \quad (2.20)$$

This group metric defines invariants on products of group generators, such as the Casimir operator. In particular, for the non-commuting operators Γ^μ that carry a space-time index conjugate to the 4-momentum, a group metric describing an invariance is given by

$$\eta_{\Gamma^\mu \Gamma^\nu}^{(EP)} = 8 \eta_{\mu\nu} \quad (2.21)$$

where $\eta_{\mu\nu}$ is the usual Minkowski metric of the Lorentz group. The Minkowski metric is thus non-trivially generated explicitly within the extended Lorentz group algebra (beyond the Lorentz invariance *implicit* in Lorentz transformations). This *group theoretic* metric can be used to develop Lorentz invariants using the operators Γ^μ , which transforms as a contravariant 4-vector operator. Its explicit use in group invariants directly connect group operations to curvilinear coordinate transformations. One should note that the group operator $\mathcal{I} \equiv \eta_{\Gamma^\mu \Gamma^\nu}^{(EP)} \Gamma^\mu \Gamma^\nu$ is not proportional to the identity matrix, or even a diagonal matrix, for general representations $\Gamma > \frac{1}{2}$. However, \mathcal{I} commutes with all generators of Lorentz group transformations, as does $\Gamma^\mu P_\mu$.

The standard Poincare group has no non-commuting operators that can be used to explicitly connect the group structure to the metric properties of space-time translations. Since the generators P_μ transform as covariant 4-vectors under arbitrary coordinate transformations (of which group transformations are a specific subset), the group structure generating the linear spinor fields is explicitly tied to curvilinear space-time dynamics through the principle of equivalence.

3 A Complete Set of Hermitian Generators

The fundamental representation of the extended Lorentz group can be developed in terms of 4×4 matrices of the group $GL(4)$. Since there are 16 Hermitian generators whose representations are 4×4 matrices, there are an additional 12 Hermitian generators in the group[6]. Of course, one of those generators is proportional to the identity matrix, and it generates a $U(1)$ internal abelian symmetry group defining a conserved hypercharge on the algebra.

One can construct a linearly independent set of 11 additional generators using the Hermitian forms of the anti-Hermitian generators $\mathbf{\Gamma}^j$ and \mathbf{K}_j given by $\mathbf{T}_j = i \mathbf{\Gamma}^j$ and $\mathbf{T}_{j+3} = i \mathbf{K}_j$, two additional independent generators given by

$$\mathbf{T}_7 = \frac{i}{2} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}, \quad \mathbf{T}_8 = \frac{1}{2} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \quad (3.22)$$

and a final set of three generators \mathbf{T}_9 , \mathbf{T}_{10} , and \mathbf{T}_{11} forming a closed representation of $SU(2)$ on the lower components:

$$\mathbf{T}_{j+8} = \frac{1}{2} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sigma_j \end{pmatrix}, \quad (3.23)$$

where $\mathbf{0}$ are 2×2 zero matrices. Although they are linearly independent of all other generators in the unified group, the set of 8 Hermitian generators \mathbf{T}_s for $s : 1 \rightarrow 8$ do not form a closed algebra independent

of the other Hermitian generators. However, of course, the group of all 15 traceless generators close within the algebra of $SU(4)$.

In the constructions that follow, the internal group symmetries will initially be developed for a massive particle state transformed 4-spinor of the form $\bar{\psi}(x) = \begin{pmatrix} \bar{\phi}_1(x)e^{i\bar{\omega}_1(x)} \\ 0 \\ 0 \\ 0 \end{pmatrix}$. While the generators $\mathbf{T}_9, \mathbf{T}_{10},$ and \mathbf{T}_{11} leaves the transformed state spinor $\bar{\psi}$ invariant, there is no combination of the generators \mathbf{T}_s for $s : 1 \rightarrow 8$ that does so. However, one can construct a set of linearly independent generators of an $SU(3)$ algebra that *can* be transformed into an internal symmetry through CKM mixing between three generations. In what follows, a straightforward set of generators for internal $SU(2)$ and $SU(3)$ symmetries will be constructed to demonstrate the unified closure of a group of 11 linearly independent hermitian generators, along with the 4 hermitian generators in the extended Lorentz group defining the representations of the linear spinor fields.

3.1 Invariance of an internal $SU(2)$ algebra

The generators $\tau_j \equiv \mathbf{T}_{j+8}$ transform under a set of internal transformations $\mathbf{M}^{(2)}$ in the space of reduced dimension that leaves the transformed spinor $\bar{\psi}$ unchanged, in the form

$$\mathbf{M}^{(2)} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^{(2)} \end{pmatrix}, \quad (3.24)$$

where $\mathbf{S}^{(2)}$ is a unitary unimodular transformation matrix in $SU(2)$, and $\mathbf{1}$ is the 2×2 identity matrix. All generators transform in this reduced dimensional subspace according to $\mathbf{G}'_r = \mathbf{M}^{(2)}\mathbf{G}_r(\mathbf{M}^{(2)})^{-1}$, preserving their group algebra. The transformed state spinor $\bar{\psi}$ is invariant under transformations involving $\mathbf{M}^{(2)}$, making it an internal invariance group for this spinor, since any $SU(2)$ rotated set of the generators could alternatively have been chosen to construct this independent subspace without altering any of the extended Lorentz group generators.

It is clear that the eigenbasis of this chosen set of independent $SU(2)$ transformations is the same as that of the generators \mathbf{I}^0 and \mathbf{J}_3 that describe (little group) invariance transformations on the particle mass states. This means that transitions that are purely induced by this $SU(2)$ symmetry will preserve internal quantum numbers within any single mass eigenstate generation of $\bar{\psi}$. However, there is no additional set of linearly independent $SU(3)$ transformations in this basis, requiring that any mix of $SU(2)$ and $SU(3)$ induced transformations necessarily involves mixing among eigenbases. This mixing will be demonstrated in the next section.

3.2 Construction of an internal SU(3) algebra

To begin, consider the following set of SU(3) generators that leave $\bar{\psi}$ invariant:

$$\begin{aligned}
\mathbf{t}_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}, & \mathbf{t}_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{2} \\ 0 & 0 & -\frac{i}{2} & 0 \end{pmatrix}, & \mathbf{t}_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}, \\
\mathbf{t}_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}, & \mathbf{t}_5 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{2} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{i}{2} & 0 & 0 \end{pmatrix}, & \mathbf{t}_6 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\mathbf{t}_7 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \mathbf{t}_8 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.
\end{aligned} \tag{3.25}$$

The generators \mathbf{t}_j define a set of internal transformations $\mathbf{M}^{(3)}$ in the space of reduced dimension of the form

$$\mathbf{M}^{(3)}(\underline{\alpha}) = \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{S}^{(3)} \end{pmatrix} = e^{i \sum \alpha^s \mathbf{t}_s}, \tag{3.26}$$

where $\mathbf{S}^{(3)}$ is a unitary unimodular transformation matrix in SU(3) and $\mathbf{0}$ is a 1×3 zero vector. The transformed state spinor $\bar{\psi}$ is invariant under transformations involving $\mathbf{M}^{(3)}$, making $\mathbf{M}^{(3)}$ an internal invariance group for the spinor. The SU(3) ‘flavor’ eigenstates will be defined using this basis.

It is important to note that the matrices \mathbf{t}_s do not form a set of linearly independent generators in the particle state representation eigenbasis for which $\mathbf{\Gamma}^0$ and \mathbf{J}_3 are diagonal (in part because there can only be 3 independent diagonal traceless generators total). A construction of independent generators in the particle state eigenbasis requires that three ‘generations’ of ‘flavor’ eigenstates be mixed from the reduced subspace. A convenient mechanism for developing the appropriate mixing is provided through general CKM matrices[7, 8] embedded within GL(4). The particular choice for mixing will be the set of all transformations on the 3×3 subspace that leaves the causal partner of $\bar{\psi}$ invariant demonstrated below:

$$\begin{aligned}
\mathbf{M}_{23} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta_{23}) & \sin(\theta_{23}) & 0 \\ 0 & -\sin(\theta_{23}) & \cos(\theta_{23}) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & \mathbf{M}_{31} &= \begin{pmatrix} \cos(\theta_{31}) & 0 & \sin(\theta_{31})e^{-i\delta_{31}} & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta_{31})e^{i\delta_{31}} & 0 & \cos(\theta_{31}) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\mathbf{M}_{12} &= \begin{pmatrix} \cos(\theta_{12}) & \sin(\theta_{12}) & 0 & 0 \\ -\sin(\theta_{12}) & \cos(\theta_{12}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & \mathbf{U}_{CKM} &\equiv \mathbf{M}_{23}\mathbf{M}_{31}\mathbf{M}_{12}.
\end{aligned} \tag{3.27}$$

Most CKM transformations of this type on the generators $\tilde{\mathbf{t}}_s \equiv \mathbf{U}_{CKM}\mathbf{t}_s\mathbf{U}_{CKM}^\dagger$ will produce a set of generators $\tilde{\mathbf{t}}_s$ satisfying the algebra of SU(3) that, along with three additional generators, form a group of 11 linearly independent hermitian generators alternative to the previous set $\{\mathbf{T}_1, \dots, \mathbf{T}_{11}\}$.

For clarity, consider the example CKM transformation from the SU(3) eigenbasis to the $\mathbf{\Gamma}^0, \mathbf{J}_3$ eigenbasis parameterized using $(\theta_{12} \rightarrow 0, \theta_{23} \rightarrow 0, \theta_{31} \rightarrow \frac{\pi}{4}, \delta_{31} \rightarrow 0)$:

$$\mathbf{U}_{CKM} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3.28)$$

The prior set of 11 linearly independent hermitian generators $\{\mathbf{T}_1, \dots, \mathbf{T}_{11}\}$ can be directly decomposed in terms of a new set of 11 linearly independent hermitian generators given by

$$\left\{ \tilde{\mathbf{t}}_1, \dots, \tilde{\mathbf{t}}_8, \tilde{\mathbf{\Delta}}_1 \equiv (\mathbf{T}_1 - \mathbf{T}_5)/2, \tilde{\mathbf{\Delta}}_2 \equiv (\mathbf{T}_2 + \mathbf{T}_4)/2, \tilde{\mathbf{\Delta}}_3 \equiv (\mathbf{T}_3 + \mathbf{T}_7)/2 \right\}.$$

The set of matrices $\{\tilde{\mathbf{t}}_1, \dots, \tilde{\mathbf{t}}_8\}$ continue to obey the same closed SU(3) algebra as the transformed generators $\{\mathbf{t}_1, \dots, \mathbf{t}_8\}$ in the internal SU(3) eigenbasis. To demonstrate their linear independence in the unified algebra of the linear spinor fields, the original set of 11 hermitian matrices are decomposed in Eqn. 3.29.

$$\begin{aligned} \mathbf{T}_1 &= -\mathbf{J}_2 - \sqrt{2}\tilde{\mathbf{t}}_2 + \sqrt{2}\tilde{\mathbf{t}}_7 + 2\tilde{\mathbf{\Delta}}_1 & \mathbf{T}_2 &= \mathbf{J}_1 - \sqrt{2}\tilde{\mathbf{t}}_1 - \sqrt{2}\tilde{\mathbf{t}}_6 + 2\tilde{\mathbf{\Delta}}_2 & \mathbf{T}_3 &= -\tilde{\mathbf{t}}_5 + \tilde{\mathbf{\Delta}}_3 \\ \mathbf{T}_4 &= -\mathbf{J}_1 + \sqrt{2}\tilde{\mathbf{t}}_1 + \sqrt{2}\tilde{\mathbf{t}}_6 & \mathbf{T}_5 &= -\mathbf{J}_2 - \sqrt{2}\tilde{\mathbf{t}}_2 + \sqrt{2}\tilde{\mathbf{t}}_7 & \mathbf{T}_6 &= -\mathbf{J}_3 - \frac{3}{2}\tilde{\mathbf{t}}_3 - \tilde{\mathbf{t}}_4 + \frac{1}{2}\tilde{\mathbf{t}}_8 \\ \mathbf{T}_7 &= \tilde{\mathbf{t}}_5 + \tilde{\mathbf{\Delta}}_3 & \mathbf{T}_8 &= -\mathbf{J}_3 - \frac{3}{2}\tilde{\mathbf{t}}_3 + \tilde{\mathbf{t}}_4 + \frac{1}{2}\tilde{\mathbf{t}}_8 & & \\ \mathbf{T}_9 &= \sqrt{2}\tilde{\mathbf{t}}_1 - \tilde{\mathbf{\Delta}}_2 & \mathbf{T}_{10} &= -\sqrt{2}\tilde{\mathbf{t}}_2 + \tilde{\mathbf{\Delta}}_1 & \mathbf{T}_{11} &= \frac{1}{2}(-\mathbf{\Gamma}^0 + \mathbf{J}_3 - \tilde{\mathbf{t}}_3 - \tilde{\mathbf{t}}_8) \end{aligned} \quad (3.29)$$

The previously discussed 3 internal SU(2) generators $\tau_j \equiv \mathbf{T}_{j+8}$ are no longer in the set of 11 linearly independent hermitian generators. One sees that any relationship between the ‘flavor’ eigenstates of linearly independent generators of an internal SU(3) symmetry and the ‘flavor’ eigenstates of linearly independent generators of an internal SU(2) symmetry, $\mathbf{\Gamma}^0$ and \mathbf{J}_3 on $\bar{\psi}$ within the confines of the extended Poincare group defining particle state representations necessarily involves CKM mixing between three ‘generations’.

The generators $\{\tilde{\mathbf{t}}_1, \dots, \tilde{\mathbf{t}}_8\}$ define an independent group of SU(3) transformations on the 4-spinors that will be denoted $\tilde{\mathcal{S}}$. Generally using this procedure for arbitrary CKM transformations, an internal SU(3) symmetry group for the transformed state spinor $\bar{\psi}$ is defined using the CKM transformation

$$\mathbf{M}^{(3)} \equiv \mathbf{U}_{CKM}^{-1} \tilde{\mathcal{S}} \mathbf{U}_{CKM}, \quad (3.30)$$

where the generators $\tilde{\mathbf{t}}_r$ of the symmetry group $\tilde{\mathcal{S}}$ must form a linearly independent set of hermitian matrices in SU(4) consistent with the extended Lorentz group transforming particle state representations within that same group space.

3.3 Transformation properties of causal fields

Pairs of quantum fields that obey microscopic causality either commute or anti-commute for space-like separations $(\vec{y} - \vec{x})$ of the space-time coordinates of those fields (i.e. outside of the light cone). The form of

a causal spinor field that has the expected properties under parity, time reversal, and charge conjugation is given by

$$\hat{\Psi}_{(\gamma)}^{(\Gamma)}(\vec{x}) \equiv \frac{1}{\sqrt{2}} \sum_{J, s_z} \int \frac{mc^2 d^3 p}{\epsilon(\mathbf{p})} \left[\frac{e^{\frac{i}{\hbar}(\mathbf{p} \cdot \mathbf{x} - \epsilon(\mathbf{p})t)}}{(2\pi\hbar)^{3/2}} \mathbf{u}_{(\gamma)}^{(\Gamma)}(\vec{p}, m, J, s_z) \hat{a}_{(\gamma)}^{(\Gamma)}(\vec{p}, m, J, s_z) + \right. \\ \left. (-)^{J+s_z} \frac{e^{-\frac{i}{\hbar}(\mathbf{p} \cdot \mathbf{x} - \epsilon(\mathbf{p})t)}}{(2\pi\hbar)^{3/2}} \mathbf{u}_{(-\gamma)}^{(\Gamma)}(\vec{p}, m, J, -s_z) \hat{a}_{(-\gamma)}^{(\Gamma)\dagger}(\vec{p}, m, J, s_z) \right], \quad (3.31)$$

in terms of the creation and annihilation operators for the particle states. The normalization has been chosen to have non-relativistic correspondence, $\frac{mc^2 d^3 p}{\epsilon(\mathbf{p})} \rightarrow d^3 p$ for $p \ll mc$. The fields are causal in that they anti-commute/commute outside of the light cone according to whether the spin $J_{max} = \Gamma$ is a half-integer or an integer, i.e.

$$\left[\hat{\Psi}_{(\gamma)}^{(\Gamma)}(\vec{y}), \hat{\Psi}_{(\gamma)}^{(\Gamma)}(\vec{x}) \right]_{\pm} = 0 \text{ for } (\vec{y} - \vec{x}) \cdot (\vec{y} - \vec{x}) > 0, \text{ where } \pm = -(-1)^{2J}. \quad (3.32)$$

Microscopic causality compels a well defined relationship of the contributions of spinor states $\mathbf{u}_{(\gamma)}^{(\Gamma)}(\vec{p}, m, J, s_z)$ to those of their causal partners $\mathbf{u}_{(-\gamma)}^{(\Gamma)}(\vec{p}, m, J, -s_z)$ in the construction of a causal field in configuration space. Under general Poincare transformations, the fields transform according to

$$\hat{U}(\Lambda, \vec{a}) \left[\hat{\Psi}_{(\gamma)}^{(\Gamma)}(\vec{x}) \right]_b \hat{U}^\dagger(\Lambda, \vec{a}) = \sum_{b'} \mathcal{D}_{bb'}^{(\Gamma)}(\Lambda^{-1}) \left[\hat{\Psi}_{(\gamma)}^{(\Gamma)}(\Lambda \vec{x} + \vec{a}) \right]_{b'}, \quad (3.33)$$

where the matrices $\mathcal{D}_{bb'}^{(\Gamma)}(\Lambda)$ form a finite-dimensional representation of the Lorentz group of transformations Λ [9].

In order to establish a relationship between a causal spinor field Ψ and the transformed spinor $\bar{\psi}$, a particular local Euclidean rotation \mathbf{R}_L will be parameterized as follows:

$$\mathbf{R}_{14} = \begin{pmatrix} \cos(\zeta_{14}) & 0 & 0 & \sin(\zeta_{14})e^{i\omega_{14}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin(\zeta_{14})e^{-i\omega_{14}} & 0 & 0 & \cos(\zeta_{14}) \end{pmatrix}, \quad \mathbf{R}_{13} = \begin{pmatrix} \cos(\zeta_{13}) & 0 & \sin(\zeta_{13})e^{i\omega_{13}} & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\zeta_{13})e^{-i\omega_{13}} & 0 & \cos(\zeta_{13}) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ \mathbf{R}_{12} = \begin{pmatrix} \cos(\zeta_{12}) & \sin(\zeta_{12})e^{i\omega_{12}} & 0 & 0 \\ -\sin(\zeta_{12})e^{-i\omega_{12}} & \cos(\zeta_{12}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{R}_L \equiv \mathbf{R}_{14}\mathbf{R}_{13}\mathbf{R}_{12} \quad (3.34)$$

where the angles and phases are chosen according to the general form

$$\Psi(x) = \mathbf{R}_L(x)\bar{\psi}(x) = \begin{pmatrix} \phi_1(x)e^{i\omega_1(x)} \\ \phi_2(x)e^{i\omega_2(x)} \\ \phi_3(x)e^{i\omega_3(x)} \\ \phi_4(x)e^{i\omega_4(x)} \end{pmatrix} = \begin{pmatrix} \phi_1(x)e^{i\omega_1(x)} \\ -\phi_1(x)\tan(\zeta_{12})\sec(\zeta_{13})\sec(\zeta_{14})e^{i\omega_2(x)} \\ -\phi_1(x)\tan(\zeta_{13})\sec(\zeta_{14})e^{i\omega_3(x)} \\ -\phi_1(x)\tan(\zeta_{14})e^{i\omega_4(x)} \end{pmatrix}, \quad (3.35)$$

with $\omega_{1s} \equiv \omega_1 - \omega_s$. The transformed spinor with the internal symmetry groups then has the form

$$\bar{\psi}(x) = \begin{pmatrix} \sqrt{(\phi_1(x))^2 + (\phi_2(x))^2 + (\phi_3(x))^2 + (\phi_4(x))^2} e^{i\omega_1(x)} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{R}_L^{-1}(x)\Psi(x). \quad (3.36)$$

Using this relationship, internal local symmetries on the general causal fields will be demonstrated in the next section.

3.4 Local gauge symmetries of linear spinor fields

Using the internal SU(3) symmetry transformation $\mathbf{M}^{(3)}$ on $\bar{\psi}$ from Eqn. 3.30, and the relationship of the transformed spinor $\bar{\psi}$ to the general causal spinor field expressed in Eqn. 3.36, a local internal SU(3) symmetry on the causal spinor field is given by

$$\mathbf{U}^{(3)}(x) = \mathbf{R}_L(x) \mathbf{M}^{(3)}(x) \mathbf{R}_L^{-1}(x) = \mathbf{R}_L(x) \mathbf{U}_{CKM}^{-1} \tilde{\mathcal{S}}(\underline{\alpha}(x)) \mathbf{U}_{CKM} \mathbf{R}_L^{-1}(x), \quad (3.37)$$

where $\alpha^s(x)$ are the eight generally locally dependent gauge group parameters of SU(3), and one representation of $\tilde{\mathcal{S}}$ is given by $\tilde{\mathcal{S}}(\underline{\alpha}) = e^{i \sum \alpha^s \tilde{\mathbf{t}}_s}$. It should be reemphasized that the set of SU(3) generators $\tilde{\mathbf{t}}_s$ on the SU(4) space must be a subset of the linearly independent generators that include those of the extended Lorentz group.

Similarly, the internal SU(2) symmetry transformation $\mathbf{M}^{(2)}$ on $\bar{\psi}$ from Eqn. 3.24 defines a local internal SU(2) symmetry on the causal spinor field given by

$$\mathbf{U}^{(2)}(x) = \mathbf{R}_L(x) \mathbf{M}^{(2)}(\underline{\theta}(x)) \mathbf{R}_L^{-1}(x), \quad (3.38)$$

where $\theta^j(x)$ are the three generally locally dependent gauge group parameters of SU(2), and $\mathbf{M}^{(2)}(\underline{\theta}) = e^{i \sum \theta^j \tau_j}$. Both (3.37) and (3.38) are local internal symmetries on the causal spinor field $\mathbf{U}(x) \Psi(x) = \Psi(x)$. However, the internal SU(2) and SU(3) symmetries do not share the same eigenbasis, since no set of linearly independent generators including the extended Lorentz group, SU(2), and SU(3) can be found. The different eigenbases are related via CKM mixing in the enlarged unified group via \mathbf{U}_{CKM} .

For any physical system with an internal symmetry group, the assignment of local space-time coordinate dependence to group transformation parameters results in a system with gauge invariance, as long as the generators for space-time translations \hat{P}_β are replaced in any Lagrangian or field equation describing the dynamics using minimal coupling $\hat{P}_\beta \rightarrow \mathbf{1} \hat{P}_\beta - \sum_r \frac{q}{c} A_\beta^r(x) \mathbf{G}_r$, where \mathbf{G}_r represents the generator of infinitesimal transformations along group parameter $\alpha^r(x)$, and $A_\beta^r(x)$ represents the gauge field. The local internal symmetry $\mathbf{U}(\underline{\alpha}(x))$ is maintained as long as the gauge fields transform according to (see reference [10] or section 3.4.1 of reference [1])

$$A_\beta^r(x : \underline{\alpha}) = \sum_s A_\beta^s(x) \oplus_s^r(\underline{\alpha}(x)) + a_\beta^r(x), \quad \partial_\beta \mathbf{U}(\underline{\alpha}(x)) = \frac{q}{\hbar c} \sum_s a_\beta^s(x) i \mathbf{G}_s \mathbf{U}(\underline{\alpha}(x)), \quad (3.39)$$

where $\mathbf{U}(\underline{\alpha}) \mathbf{G}_s \mathbf{U}^{-1}(\underline{\alpha}) \equiv \sum_r \oplus_s^r(\underline{\alpha}) \mathbf{G}_r$, and the functions $a_\beta^s(x)$ are related to the derivative of the group parameters. The gauge group topology of the local mapping in space-time of the group structure functions determines the monopole structure of the sources of the gauge interactions (see section 4.2 of [1]). The inclusion of geometrodynamics occurs via the principle of equivalence, with a replacement of the operations

in locally flat space-time using covariance: $\Gamma^\mu P_{\xi^\mu} \rightarrow \Gamma^\mu \frac{\partial x^\beta}{\partial \xi^\mu} P_\beta$, where ξ^μ are locally flat coordinates with conjugate momenta P_{ξ^μ} . The operators P_{ξ^μ} are the generators in the extended Poincare algebra.

4 Conclusions

General formulations of scattering theory that are unitary, maintain quantum linearity in space-time translation generators, have positive definite energies, and have straightforward cluster decomposition properties, can be constructed in a straightforward manner using linear spinor fields. The fundamental representation of linear spinor fields unifies a set of internal local symmetries including a U(1) symmetry along with 11 additional hermitian generators that can represent a linearly independent SU(2) symmetry or a linearly independent SU(3) symmetry, but not both. The eigenbasis of the linearly independent SU(3) symmetry has been shown to be related to that of the SU(2) symmetry via a CKM transformation mixing the symmetries in SU(4). Internal local gauge symmetries for SU(2) and SU(3) on causal spinor fields have been demonstrated.

The geometrodynamics of general relativity is directly incorporated in the field equation satisfied by the linear spinor fields through the principle of equivalence. Furthermore, group algebraic invariants explicitly include the Minkowski metric as calculated using the non-abelian algebra of generators that carry space-time indices, extending interior group structure to the dynamics of general coordinate transformations. The piece of the group algebra that connects the group structure to metric gravitation via the equivalence principle (the algebra of $\hat{\Gamma}^\beta$) necessitates the inclusion of an additional group operator that generates affine parameter translations for massless particles. This allows dynamic mixing of massless particles in a manner not allowed by the standard formulations of Dirac or Majorana. On-going work is examining the extent that the present neutrino mixing phenomenology can be modeled considering neutrinos as *massless* spinor fields of differing transverse mass. Future work will examine the quantum number flows via the self-adjoint $(\gamma) = 0$ degenerate eigenstates of $\hat{\Gamma}^\mu \hat{P}_\mu$ for the $\Gamma = 1$ representation of linear spinor fields that can mix with massless vector particles. This representation contains a self-adjoint scalar particle, a self-adjoint vector particle, another vector particle, and its adjoint vector particle.

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References

- [1] J. Lindesay, *Foundations of Quantum Gravity*, Cambridge University Press (2013), ISBN 978-1-107-00840-3.
- [2] J.V. Lindesay, A.J. Markevich, H.P. Noyes, and G. Pastrana, Phys. Rev. D33, 2339 (1986).
- [3] M. Alfred, P. Kwizera, J.V. Lindesay, and H.P. Noyes, A Non-Perturbative, Finite Particle Number Approach to Relativistic Scattering Theory, SLAC-PUB-8821, hep-th/0105241 (2001).
- [4] P.A.M. Dirac, Proc. Roy. Soc. (London), A117, 610 (1928); *ibid*, A118, 351 (1928).
- [5] J.D. Bjorken and S.D. Drell, *Relativistic Quantum Mechanics*, McGraw-Hill, New York (1964).
- [6] J. Lindesay, “Linear Spinor Fields in Relativistic Dynamics”, arXiv:1312.0541 [hep-th] (2013) 13 pages.
- [7] N. Cabibbo, “Unitary Symmetry and Leptonic Decays”. *Physical Review Letters* **10** (12): 531-533 (1963). doi:10.1103/PhysRevLett.10.531 .
- [8] M. Kobayashi and T. Maskawa, “CP-Violation in the Renormalizable Theory of Weak Interaction”. *Progress of Theoretical Physics* **49** (2): 652-657 (1973). doi:10.1143/PTP.49.652 .
- [9] S. Weinberg, *The Quantum Theory of Fields*, Cambridge University Press, Cambridge (1995).
- [10] H. Morrison and J. Lindesay, “The Geometry of Quantum Flow”. *Mathematical Analysis of Physical Systems*, pp 135-167, R. Mickens, editor. Van Nostrand Reinhold, Co., New York (1985)